Mathematics 222B Lecture 21 Notes

Daniel Raban

April 12, 2022

1 The Vector Field Method for Dispersive Decay for the Wave Equation

1.1 Motivation for the vector field method

Today, we will continue discussing dispersive decay for the wave equation

$$\begin{cases} \Box\phi = 0 & \text{in } \mathbb{R}^{1+d} \\ (\phi, \partial_t\phi)|_{t=0} = (g, h). \end{cases}$$

Last time, we applied oscillatory integral techniques to the Fourier-analytic representation of the (frequency localized) fundamental solution. This led to the following dispersive inequality.

Theorem 1.1 (Dispersive inequality)**.** *For a solution ϕ to the wave equation,*

$$\|\phi(t)\|_{L^\infty} \lesssim t^{-\frac{d-1}{2}}(\|g\|_{B_1^{\frac{d+1}{2},1}} + \|h\|_{B_1^{\frac{d-1}{2},1}}).$$

This is the starting point for many estimates that are useful for **semilinear wave equations**, equations of the form $\Box\phi = N(\phi, \nabla\phi)$ with principal term $= \Box\phi$. But many equations of interest may have quasilinear nonlinearity ($g^{\mu,\nu}(\phi, \nabla\phi)\partial_\mu\partial_\nu\phi$) or just $g^{\mu,\nu}(t,x) \neq m^{\mu,\nu}$, for which the previous approach is harder to generalize.

Today, we will cover the vector field method, introduced by Klaineman in the 80s. This is a purely physical space method (as opposed to the Fourier analytic method above).[1] The motivating question is: How do we derive pointwise bounds for $\nabla_{t,x}\phi$ from the energy method?

Step 1: The energy estimate tells us that

$$E[\phi](t) = \int \frac{1}{2}(\partial_t\phi)^2 + \frac{1}{2}|D\phi|^2\,dx$$

[1]In general, Fourier analytic methods work best for constant coefficient, linear equations because when multiplication is involved, it becomes convolution, which can get messy.

is conserved. We can express this as a bound

$$\|\nabla_{t,x}\phi(t)\|_{L^2} \lesssim \|\nabla_{t,x}\phi|_{t=0}\|_{L^2}.$$

Step 2: Notice that if $\Box\phi = 0$, then any derivative satisfies

$$\Box(\partial_\mu\phi) = \partial_\mu\Box\phi = 0.$$

The energy estimate for the derivative then tells us that

$$\|\nabla_{t,x}D^\alpha\phi(t)\|_{L^2} \lesssim_\alpha \|\nabla_{t,x}D^\alpha\phi|_{t=0}\|_{L^2}.$$

Step 3: For $s > \frac{d}{2}$, we can use the Sobolev inequality to get

$$\begin{aligned}\|\nabla_{t,x}\phi(t)\|_{L^\infty_x} &\lesssim \|\nabla_{t,x}\phi(t)\|_{H^s_x} \\ &\lesssim \|\nabla_{t,x}\phi|_{t=0}\|_{H^s_x}.\end{aligned}$$

The basic idea of Klaineman's method was to follow this format to prove dispersive decay. The goal is to derive a pointwise estimate of the form

$$\|\nabla_{t,x}\phi(t)\|_{L^\infty_x} \lesssim t^{-\frac{d-1}{2}}(\text{Initial data}).$$

What parts of the approach should we modify? The first key idea is to modify step 2 of the argument above. The key property is that if $[\partial_\mu, \Box] = 0$, then $\Box\phi = 0$ implies $\Box\partial_\mu\phi = 0$. Thinking about this more geometrically, consider the translation operator $\phi \mapsto \mathcal{T}_{x^\mu,h}\phi = \phi((t,x) + he_\mu)$, where

$$\partial_\mu\phi = \frac{d}{dh}\mathcal{T}_{c^\mu,h}\phi|_{h=0},$$

so $\mathcal{T}_{c^\mu,h}$ is the infinitesimal generator for ∂_μ. The important thing to notice is that $\mathcal{T}_{x^\mu,h}$ is a symmetry for $\Box$:

$$\Box\mathcal{T}_{x^\mu,h}\phi = \mathcal{T}_{x^\mu,h}\Box\phi.$$

This process can be applied to any symmetries of $\Box$!

1.2 Symmetries of the d'Alembertian

Recall that the symmetries of $\Box$ are the linear symmetries $\mathbb{R}^{1+d} \to \mathbb{R}^{1+d}$ that preserve $m(v,w) = m^{\mu,\nu}v_\mu w_\nu$, where $m = \operatorname{diag}(-1,1,1,\dots,1)$. This means we want to look for matrices L_t such that

$$m^{\mu,\nu}(L_t)_\mu^{\mu'}(L_t)_\nu^{\nu'} = m^{\mu,\nu}.$$

If we assume that $L_0 = I$, then differentiating in t gives (denoting $\ell = \frac{d}{dt}L_t|_{t=0}$)

$$m^{\mu,\nu'}\ell_\mu^{\mu'} + m^{\mu',\nu}\ell_\nu^{\nu'} = 0.$$

If we define $\widetilde{\ell}^{\mu,\nu} = m^{\mu,\nu'}\ell_\mu^{\mu'}$, then we get $\widetilde{\ell}^{\nu',\mu'} + \widetilde{\ell}^{\mu',\nu'} = 0$.

The symmetries of $\Box$ turn out to be compositions of the following:

- Translations $\mathcal{T}_{x^\mu,h}$
- Rotations

$$\mathcal{R}_{x^1,x^2,h} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos h & -\sin h & 0 \\ 0 & \sin h & \cos h & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

- Lorentz boosts

$$\mathcal{L}_{x^1,h} = \begin{bmatrix} \frac{1}{\sqrt{1-h^2}} & -\frac{h}{\sqrt{1-h^2}} & 0 \\ -\frac{h}{\sqrt{1-h^2}} & \frac{1}{\sqrt{1-h^2}} & 0 \\ 0 & 0 & I \end{bmatrix}$$

The infinitesimal generators (meaning operators $\frac{d}{dh}\mathcal{S}_h(\cdot)|_{h=0}$ are

$$\Omega_{1,2} = x^1\partial_{x^2} - x^2\partial_{x^1}.$$

$$L_1 = x^1\partial_t + t\partial x^1$$

and the corresponding generators for the other indices. Observe that

$$[\partial_\mu, \Box] = [\Omega_{j,k}, \Box] = [L_j, \Box] = 0.$$

Also consider the scaling operator

$$\mathcal{S}_h\phi = \phi(t/h, t/x).$$

with intinitesimal generator

$$S\phi = -\frac{d}{dh}\mathcal{S}_h\phi|_{h=1} = (t\partial_t + x\partial_x)\phi.$$

This is not a symmetry of $\Box$, because

$$\begin{aligned} \Box\mathcal{S}_h\phi &= \Box\phi(t/h, x/h) \\ &= \frac{1}{h^2}(\Box\phi)(t/h, x/h) \\ &= \frac{1}{h^2}\mathcal{S}_h(\Box\phi). \end{aligned}$$

However, if $\Box\phi = 0$, then $\Box\mathcal{S}_h\phi = 0$. This is a reflection of the fact that

$$S\Box = S\Box - 2\Box,$$

where the -2 represents the homogeneity of $\Box$, a second order operator.

For $\Gamma \in \{\partial_0, \dots, \partial_d, \Omega_{1,2}, \dots, \Omega_{(d-1),d}, L_1, \dots, L_d, S\}$, labeled in order as $\Gamma_1, \Gamma_2, \dots, \Gamma_K$, we let

$$\Gamma^\alpha\phi = \Gamma_1^{\alpha_1}\cdots\Gamma_K^{\alpha_K}\phi, \qquad \alpha \in \mathbb{R}^K.$$

1.3 Bounds on commuting symmetries with derivatives

Our discussion has told the the following:

Lemma 1.1. *If $\Box\phi = 0$, then $\Box\Gamma^\alpha\phi = 0$ for all α.*

The energy estimate gives the following.

Corollary 1.1.

$$\|\nabla_{t,x}\Gamma^\alpha\phi(t)\|_{L^2} \lesssim_\alpha \|\nabla_{t,x}\Gamma^\alpha\phi|_{t=0}\|_{L^2}.$$

Lemma 1.2. *Given any smooth function ψ,*

$$|\Gamma^\alpha\nabla_{t,x}\psi| \lesssim \sum_{\beta:|\beta|\leq|\alpha|} |\nabla_{t,x}\Gamma^\beta\psi|.$$

Here is the proof of the lemma:

Proof. When $\Gamma \in \partial_0, \dots, \partial_\mu$, there is nothing to do. When $\Gamma \in \{\Omega, L, S\}$, then $[\Gamma, \partial_{x^\mu}] = c^\nu_{\mu,\Gamma}\partial_{x^\nu}$; we can argue this by checking the generators or by claiming that these vector fields form a Lie algebra, so we get information about the Lie bracket. We complete the argument by induction. $\Box$

Corollary 1.2. *Fix s.*

$$\sum_{\alpha:|\alpha|\leq s} \|\Gamma^\alpha\nabla_{t,x}\psi(t)\|_{L^2} \lesssim \sum_{\alpha:|\alpha|\leq s} \|\nabla_{t,x}\Gamma^\alpha\phi|_{t=0}\|_{L^2}.$$

1.4 The Klaineman-Sobolev inequality and proof of the dispersive estimate

The second key idea is to modify step 3, where we used the Sobolev inequality. We first need to understand what control Γ gives us.

Define $\Omega_{\mu,\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$, where

$$x_\mu = x^\nu m_{\mu,\nu} = \begin{cases} -t & \mu = 0 \\ x^j & m = h \in \{1, \dots, d\}. \end{cases}$$

If we have $\Omega_{j,k}$ as before, then $L_j = \Omega_{j,0}$.

Lemma 1.3.

$$(t^2 - |x|^2)\partial_\mu = x_\mu S - x^\nu\Omega_{\mu,\nu}\frac{x^\nu}{|x|}L_\nu.$$

Proof. Observe that

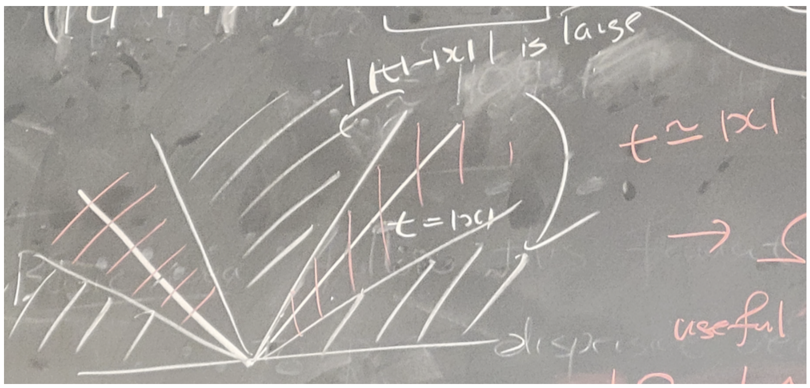
$$\begin{aligned} x^\nu \Omega_{\mu,\nu} &= x^\nu (x_\mu \partial_\nu - x_\nu \partial_\mu) \\ &= x_\mu \underbrace{x^\nu \partial_\nu}_{S} - \underbrace{x^\nu x_\nu}_{(-t^2+|x|^2)} \partial_\mu . \end{aligned}$$

□

This means that

$$(|t| - |x|)\partial_\mu = \underbrace{\frac{x_\mu}{|t|+|x|}}_{\le 1} S - \underbrace{\frac{x^\nu}{|t|+|x|}}_{\le 1} \Omega_{\mu,\nu}.$$

Away from the cone $t = |x|$, we get control of the derivatives.



In the region where $t \simeq |x|$, the rotation vector fields $\Omega_{j,k}$ are useful. The size of these rotation vector fields is $|\Omega_{j,k}| \simeq |x|$. We control all angular derivatives ($d-1$ many directions) with weight $|x| \simeq t$; this is why we get $\frac{d-1}{2}$ instead of $\frac{d}{2}$ in the dispersive estimate.

The analytic key to this method is the following inequality.

Theorem 1.2 (Klaineman-Sobolev inequality)**.** *Let ψ be a nice function, and let $s > \frac{d}{2}$. Then for $t > 0$,*

$$|\psi(t,x)| \lesssim \frac{1}{(1+v)^{\frac{d-1}{2}}(1+|u|)^{1/2}} \sum_{|\alpha| \le s} \|\Gamma^\alpha \psi\|_{L^2},$$

where $v = t - |x|$ and $u = t - |x|$.

If we apply this theorem to $\psi = \nabla_{t,x}\phi$, we get

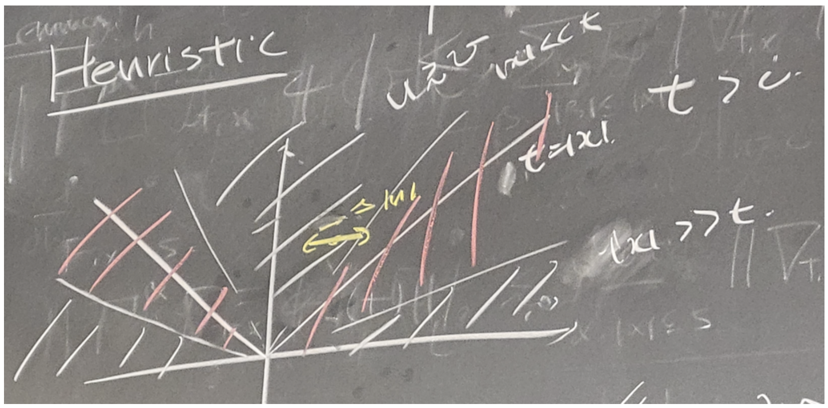
Corollary 1.3.

$$\begin{aligned} |\nabla_{t,x}\phi| &\lesssim \frac{1}{(1+v)^{\frac{d-1}{2}}(1+|u|)^{1/2}} \sum_{|\alpha| \le s} \|\Gamma^\alpha \nabla_{t,x}\phi(t)\|_{L^2} \\ &\le \frac{1}{(1+v)^{\frac{d-1}{2}}(1+|u|)^{1/2}} \sum_{|\alpha| \le s} \|\nabla_{t,x}\Gamma^\alpha \phi|_{t=0}\|_{L^2} \end{aligned}$$

Here, the factor in front is $\lesssim \frac{1}{(1+t)^{\frac{d-1}{2}}}$, so we have something a little better than our original bound.

Here is the idea behind proving the Klaineman-Sobolev inequality.

Proof. The key heuristic is that Γ gives control of $|u|\partial_{\mu,x}$. Now decompose the space into regions where $|x| \ll t$ and $x \simeq t$, and $|x| \gg t$.



Then let $w \simeq \frac{1}{1+|u|)^{d/2}}$. When $|u| \lesssim 1$, the the usual Soboolev inequality works. Otherwise, if $|u| \gtrsim 1$, then $w \simeq \frac{1}{|u|^{d/2}}$.

Lemma 1.4 (Rescaled Sobolev)**.**

$$|\psi(x)| \lesssim \frac{1}{u^{d/2}} \sum_{|\alpha| \le s} \||u|^{|\alpha|}\partial^\alpha \psi\|_{L^2(B_{|u|}(x))}$$

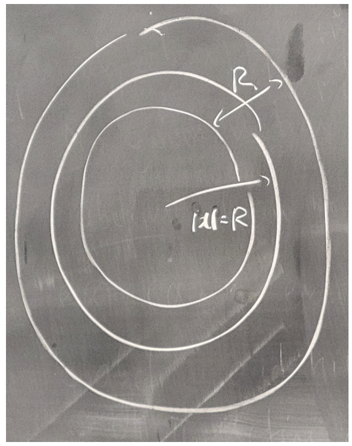
Proof. This follows from rescaling the Sobolev inequality on the unit ball $B_1(0)$. □

When t and $|x|$ are comparable, the weight $w \simeq \frac{1}{(1+v)^{\frac{d-1}{2}}}$. If $|v| \lesssim 1$, the usual Sobolev inequality works. If $|v| \gtrsim 1$, then $w \simeq \frac{1}{v^{\frac{d-1}{2}}} \simeq \frac{1}{|x|^{\frac{d-1}{2}}}$. The final lemma we use is this:

Lemma 1.5 (Rescaled Sobolev on an annulus)**.**

$$|\psi(x)| \lesssim \frac{1}{R^{\frac{d-1}{2}}} \sum_{\alpha,\beta:|\alpha|+|\beta| \le s} \left(\int_{A_R} |\partial_r^\alpha \Omega_x^\beta \psi|^2 \, dx \right)^{1/2},$$

where $A_R = \{||x| - R| \leq cR\}$.



Here, $R^{\frac{d-1}{2}}$ responds to the angular directions that Ω_x^β has control over. $\square$